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SOME RELATIONSHIPS BETWEEN SUBCLASSES OF UNIVALENT ANALYTIC FUNCTIONS INVOLVING THE WRIGHT FUNCTION

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Abstract: The aim of this article is to establish correlations between different categories of analytic univalent functions using a specific convolution operator defined by the Wright function. More specifically, we explore these correlations among the classes of analytic univalent functions $k-\mathscr{UCV}^*(\beta)$, $k-\mathscr{S}_p^*(\beta)$, $\mathscr{R}(\beta)$, $\mathscr{R}^\tau(A,\ B)$, $k-\mathscr{PUCV}^*(\beta)$ and $k-\mathscr{PS}_p^*(\beta)$ in the open unit disc $\mathbb{U}$.

Keywords and Phrases: Analytic, univalent functions, uniformly convex and starlike functions, Wright function.

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1. Introduction

Let $\mathscr{A}$ represent the set of all analytic functions within the open unit disk

$$\mathbb{U} := \{z \in \mathbb{C} \ : \ |z| < 1\}$$

with the normalization $f(0) = 0$ and $f'(0) = 1$, we denote by $\mathcal{S}$ the subset of $\mathcal{A}$ comprising functions that are univalent in $\mathbb{U}$. The functions belonging to the class $\mathcal{S}$ can be expressed through a power series expansion centered at the origin in the following manner:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Porwal and Dixit [2, 12] introduced the families $k - \mathcal{UCV}^*(\beta)$ and $k - \mathcal{S}_p^*(\beta)$ as:

A function f of the form (1.1) is classified as being in the class $k - \mathcal{UCV}^*(\beta)$ if it satisfies the following condition

$$\Re\left(1 + \left(1 + ke^{i\phi}\right)\frac{zf''(z)}{f'(z)}\right) < \beta, \tag{1.2}$$

where $0 \leq k < \infty$, $\phi \in \mathbb{R}$ and $1 < \beta \leq \dfrac{4+k}{3}$.

A function f of the form (1.1) is said to be in the class $k - \mathcal{S}_p^*(\beta)$ if it satisfies the following condition

$$\Re\left(\left(1 + ke^{i\phi}\right)\frac{zf'(z)}{f(z)} - ke^{i\phi}\right) < \beta, \tag{1.3}$$

where $0 \leq k < \infty$, $\phi \in \mathbb{R}$ and $1 < \beta \leq \dfrac{4+k}{3}$.

Further, let $\mathcal{V}$ be the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.4}$$

Let

$$k - \mathcal{PUCV}^*(\beta) = k - \mathcal{UCV}^*(\beta) \cap \mathcal{V} \tag{1.5}$$

and

$$k - \mathcal{PS}_p^*(\beta) = k - \mathcal{S}_p^*(\beta) \cap \mathcal{V} \tag{1.6}$$

It is worthy to note that for $k = 0$ the classes $k - \mathcal{UCV}^*(\beta)$, $k - \mathcal{S}_p^*(\beta)$, $k - \mathcal{PUCV}^*(\beta)$ and $k - \mathcal{PS}_p^*(\beta)$ are reduce to the classes $\mathcal{L}(\beta)$, $\mathcal{M}(\beta)$, $\mathcal{U}(\beta)$ and $\mathcal{V}(\beta)$ respectively (see [20]).

In 1995, Dixit and Pal [1] introduce the class $\mathscr{R}^{\tau}(A,\ B)$ consisting of functions $f(z)$ of the form (1.1) which satisfy the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau - B(f'(z)-1)}\right| < 1, \qquad \tau \in \mathbb{C}\setminus\{0\}, \quad -1 \leq B < A \leq 1, \quad z \in \mathbb{U}. \tag{1.7}$$

Also, $\mathscr{R}(\beta)$ denote the subclass of $\mathscr{A}$ consisting of functions $f(z)$ of the form (1.4) which satisfy the condition

$$\Re\ (f'(z)) < \beta, \qquad 1 < \beta \leq 2, \qquad z \in \mathbb{U}. \tag{1.8}$$

In 1933, Wright [22] introduced a special function, which is named as Wright function (see also [5])and is defined as

$$\mathscr{W}_{\lambda,\,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(n\,\lambda+\mu)}, \qquad \lambda > -1, \quad \mu \in \mathbb{C}, \tag{1.9}$$

where $\Gamma(\cdot)$ stands for the usual Gamma function. The series (1.9) is absolutely convergent for all $z \in \mathbb{C}$, while for $\lambda = -1$ this is absolutely convergent in $\mathbb{U}$. Also, Wright [22] shown that (1.9) is an entire function for $\lambda > -1$. The applications of Wright function and its generalization were used in partitions of natural numbers, integral transform, differential equations, wave equations etc. Noteworthy contributions in this direction may be found in [4, 6-8, 10, 11, 13, 14, 16, 17, 19].

Let $\mathbb{W}$ represent the normalized Wright functions defined by

$$\mathbb{W}_{\lambda,\,\mu}(z) = \Gamma(\mu)\, z\mathscr{W}_{\lambda,\,\mu}(z) := \sum_{n=0}^{\infty} \frac{\Gamma(\mu) z^{n+1}}{n!\,\Gamma(n\,\lambda+\mu)}, \qquad \lambda > -1, \quad \mu > 0, \tag{1.10}$$

for all $z \in \mathbb{U}$. The analytical and geometrical properties of normalized Wright function were studied by [7, 11, 13].

The convolution (or, Hadamard product) of two power series $f(z)$ of the form (1.1) and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

is given by

$$(f * g)(z) = f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \qquad z \in \mathbb{U}.$$

Now, we consider a linear operator $\Omega(\lambda,\ \mu)\ :\ \mathscr{A}\to\mathscr{A}$ defined by

$$\Omega(\lambda,\ \mu)f(z)=\mathbb{W}_{\lambda,\,\mu}(z)*f(z):=z+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)}\frac{a_n z^n}{(n-1)!}. \tag{1.11}$$

The utilization of special functions such as confluent hypergeometric functions, Gauss hypergeometric functions, generalized hypergeometric functions, Mittag-Leffler functions, Bessel functions, and generalized Bessel functions presents intriguing research avenues in Geometric Function Theory. One can refer [4, 6, 10, 14-19] for details. In this current paper, we establish correlations among the classes $k-\mathscr{UCV}^*(\beta)$, $k-\mathscr{S}_p^*(\beta)$, $\mathscr{R}(\beta)$, $\mathscr{R}^\tau\left(A,\ B\right)$, $k-\mathscr{PUCV}^*(\beta)$ and $k-\mathscr{PS}_p^*(\beta)$. We associated with normalized Wright function.

2. Preliminary Results

To prove our main results we shall require the following lemmas:

Lemma 2.1. [12] *A function $f(z)\in\mathscr{S}$ of the form (1.1) and*

$$\sum_{n=2}^{\infty}n[n+nk-k-\beta]|a_n|\leq\beta-1,$$

then $f\in k-\mathscr{UCV}^*(\beta)$.

Lemma 2.2. [12] *A function $f(z)\in\mathscr{S}$ of the form (1.1) and*

$$\sum_{n=2}^{\infty}(n+nk-k-\beta)|a_n|\leq\beta-1,$$

then $f\in k-\mathscr{S}_p^*(\beta)$.

Lemma 2.3. [1] *If $f\in\mathscr{R}^\tau\left(A,\ B\right)$ is of the form (1.1), then*

$$|a_n|\leq\frac{(A-B)\,|\tau|}{n},\qquad n\in\mathbb{N}\setminus\{1\}.$$

The result is sharp.

Lemma 2.4. [21] *If $f\in\mathscr{R}\left(\beta\right)$ is of the form (1.1), then*

$$|a_n|\leq\frac{\beta-1}{n},\qquad n\in\mathbb{N}\setminus\{1\}.$$

Lemma 2.5. [12] *Let $f(z)\in\mathscr{S}$ be of the form (1.4) and $f\in k-\mathscr{PUCV}^*(\beta)$. Then*

$$|a_n|\leq\frac{\beta-1}{n(n+nk-k-\beta)}.$$

Lemma 2.6. [12] *Let $f(z) \in \mathscr{S}$ be of the form (1.4) and $f \in k - \mathscr{PS}_p^*(\beta)$. Then*

$$|a_n| \leq \frac{\beta - 1}{n + nk - k - \beta}.$$

Lemma 2.7. [3] *For all $\lambda \geq 0$ and $\mu > 0$, we have*

(a) $\sum\limits_{n=0}^{\infty} \dfrac{\Gamma(\mu)}{(n+1)!\,\Gamma((n+1)\lambda + \mu)} = \mathbb{W}_{\lambda,\,\mu}(1) - 1.$

(b) $\sum\limits_{n=0}^{\infty} \dfrac{\Gamma(\mu)}{n!\,\Gamma((n+1)\lambda + \mu)} = \mathbb{W}'_{\lambda,\,\mu}(1) - \mathbb{W}_{\lambda,\,\mu}(1).$

(c) $\sum\limits_{n=0}^{\infty} \dfrac{\Gamma(\mu)}{(n-1)!\,\Gamma((n+1)\lambda + \mu)} = \mathbb{W}''_{\lambda,\,\mu}(1) - 2\mathbb{W}'_{\lambda,\,\mu}(1) + 2\mathbb{W}_{\lambda,\,\mu}(1).$

(d) $\sum\limits_{n=0}^{\infty} \dfrac{\Gamma(\mu)}{(n-2)!\,\Gamma((n+1)\lambda + \mu)} = \mathbb{W}'''_{\lambda,\,\mu}(1) - 3\mathbb{W}''_{\lambda,\,\mu}(1) + 6\mathbb{W}'_{\lambda,\,\mu}(1) - 6\mathbb{W}_{\lambda,\,\mu}(1).$

3. Inclusion Relations

In our first theorem of this section we obtain an inclusion relation between the classes $\mathscr{R}^\tau(A,\ B)$ and $k - \mathscr{UCV}^*(\beta)$.

Theorem 3.1. *If $\lambda,\ \mu > 0$, for some k $(0 \leq k < \infty)$, β $(1 \leq \beta < \frac{3+k}{2})$, the function $f \in \mathscr{R}^\tau(A,\ B)$ and the inequality*

$$(A - B)|\tau| \left[(k+1)\mathbb{W}'_{\lambda,\,\mu}(1) - (k+\beta)\mathbb{W}_{\lambda,\,\mu}(1) + \beta - 1\right] \leq \beta - 1 \tag{3.1}$$

is satisfied, then $\Omega(\lambda,\ \mu)f \in k - \mathscr{UCV}^*(\beta)$, where $k - \mathscr{UCV}^*(\beta)$ and $\mathscr{R}^\tau(A,\ B)$ are respectively, defined by (1.2) *and* (1.7)*, and $\Omega(\lambda,\ \mu)$ is the operator defined by* (1.11).

Proof. Since

$$\Omega(\lambda,\ \mu)f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma((n-1)\lambda + \mu)} \frac{a_n z^n}{(n-1)!}\ .$$

To prove that $\Omega(\lambda,\ \mu)f \in k - \mathscr{UCV}^*(\beta)$, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[n + nk - k - \beta] \frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda + \mu)}\ \frac{1}{(n-1)!}|a_n| \leq \beta - 1.$$

Now,

$$\begin{aligned}
&\sum_{n=2}^{\infty} n[n+nk-k-\beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}|a_n| \\
&\leq (A-B)|\tau| \sum_{n=2}^{\infty}\left[[(n-1)(1+k)-(\beta-1)]\ \frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}\right] \\
&= (A-B)|\tau| \left\{(1+k)\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-2)!} - (\beta-1)\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}\right\} \\
&= (A-B)|\tau| \left\{(1+k)\sum_{n=0}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\ \frac{1}{n!} - (\beta-1)\sum_{n=0}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\ \frac{1}{(n+1)!}\right\} \\
&= (A-B)|\tau| \left\{(1+k)\left[\mathbb{W}'_{\lambda,\,\mu}(1)-\mathbb{W}_{\lambda,\,\mu}(1)\right] - (\beta-1)\left[\mathbb{W}_{\lambda,\,\mu}(1)-1\right]\right\} \\
&= (A-B)|\tau| \left\{(1+k)\mathbb{W}'_{\lambda,\,\mu}(1) - (k+\beta)\mathbb{W}_{\lambda,\,\mu}(1)+\beta-1\right\} \\
&\leq \beta-1,
\end{aligned}$$

based on the provided hypothesis. This concludes the proof of Theorem 3.1.

Theorem 3.2. *If $\lambda,\ \mu>0$, for some k $(0\leq k<\infty)$, β_1 $(1<\beta_1\leq 2)$, β $(1\leq\beta<\frac{3+k}{2})$, the function $f\in\mathscr{R}^\tau(\beta_1)$ and the inequality*

$$(\beta_1-1)\left[(k+1)\mathbb{W}'_{\lambda,\,\mu}(1)-(k+\beta)\mathbb{W}_{\lambda,\,\mu}(1)+\beta-1\right]\leq\beta-1 \tag{3.2}$$

is satisfied, then $\Omega(\lambda,\ \mu)f\in k-\mathscr{UCV}^*(\beta)$, where $k-\mathscr{UCV}^*(\beta)$ is defined by (1.2), and $\Omega(\lambda,\ \mu)$ is the operator defined by (1.11).

Proof. Since

$$\Omega(\lambda,\ \mu)f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)}\frac{a_nz^n}{(n-1)!}\ .$$

To prove that $\Omega(\lambda,\ \mu)f\in k-\mathscr{UCV}^*(\beta)$, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[n+nk-k-\beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}|a_n|\leq\beta-1.$$

Now,

$$\begin{aligned}
&\sum_{n=2}^{\infty} n[n+nk-k-\beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\,\frac{1}{(n-1)!}|a_n| \\
&\le (\beta_1-1)\sum_{n=2}^{\infty}\left[[(n-1)(1+k)-(\beta-1)]\,\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\,\frac{1}{(n-1)!}\right] \\
&= (\beta_1-1)\left\{(1+k)\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\,\frac{1}{(n-2)!} - (\beta-1)\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\,\frac{1}{(n-1)!}\right\} \\
&= (\beta_1-1)\left\{(1+k)\sum_{n=0}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\,\frac{1}{n!} - (\beta-1)\sum_{n=0}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\,\frac{1}{(n+1)!}\right\} \\
&= (\beta_1-1)\left\{(1+k)\left[\mathbb{W}'_{\lambda,\,\mu}(1)-\mathbb{W}_{\lambda,\,\mu}(1)\right]-(\beta-1)\left[\mathbb{W}_{\lambda,\,\mu}(1)-1\right]\right\} \\
&= (\beta_1-1)\left\{(1+k)\mathbb{W}'_{\lambda,\,\mu}(1)-(k+\beta)\mathbb{W}_{\lambda,\,\mu}(1)+\beta-1\right\} \\
&\le \beta-1,
\end{aligned}$$

based on the provided hypothesis. This concludes the proof of Theorem 3.2.

Theorem 3.3. *If* $\lambda,\ \mu>0$, *for some* k $(0\le k<\infty)$, β $(1\le\beta<\dfrac{3+k}{2})$, *the function* $f\in k-\mathscr{PUCV}^*(\beta)$ *and the inequality* $\mathbb{W}_{\lambda,\,\mu}(1)-1\le 1$, *then* $\Omega(\lambda,\ \mu)f\in k-\mathscr{UCV}^*(\beta)$, *where* $k-\mathscr{UCV}^*(\beta)$ *and* $k-\mathscr{PUCV}^*(\beta)$ *are respectively, defined by* (1.2) *and* (1.5), *and* $\Omega(\lambda,\ \mu)$ *is the operator defined by* (1.11).

Proof. Let $f\in k-\mathscr{PUCV}^*(\beta)$, where f is of the form (1.4). Since

$$\Omega(\lambda,\ \mu)f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)}\frac{a_nz^n}{(n-1)!}\ .$$

To prove that $\Omega(\lambda,\ \mu)f\in k-\mathscr{UCV}^*(\beta)$, in view of Lemma 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[n+nk-k-\beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\,\frac{1}{(n-1)!}|a_n|\le\beta-1.$$

Now, in view of Lemma 2.5, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n[n+nk-k-\beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\,\frac{1}{(n-1)!}|a_n| \quad &\le(\beta-1)\sum_{n=2}^{\infty}\left[\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\,\frac{1}{(n-1)!}\right] \\
&=(\beta-1)\sum_{n=0}^{\infty}\left[\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\,\frac{1}{(n+1)!}\right] \\
&=(\beta-1)\left[\mathbb{W}_{\lambda,\,\mu}(1)-1\right] \\
&\le\beta-1,
\end{aligned}$$

which is true for all $\lambda,\ \mu > 0$. This completes the proof of Theorem 3.3.

Theorem 3.4. *If $\lambda,\ \mu > 0$, for some k $(0 \le k < \infty)$, β $(1 \le \beta < \frac{4+k}{3})$, the function $f \in k - \mathscr{PS}_p^*(\beta)$ and the inequality*

$$\mathbb{W}'_{\lambda,\,\mu}(1) - 1 \le 1 \tag{3.3}$$

is satisfied, then $\Omega(\lambda,\ \mu)f \in k - \mathscr{UCV}^*(\beta)$, where $k - \mathscr{UCV}^*(\beta)$ and $k - \mathscr{PS}_p^*(\beta)$ are respectively, defined by (1.2) and (1.6), and $\Omega(\lambda,\ \mu)$ is the operator defined by (1.11).

Proof. Let $f \in k - \mathscr{PS}^*(\beta)$, where f is of the form (1.4). Since

$$\Omega(\lambda,\ \mu)f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma((n-1)\lambda + \mu)} \frac{a_n z^n}{(n-1)!}\ .$$

To prove that $\Omega(\lambda,\ \mu)f \in k - \mathscr{UCV}^*(\beta)$, in view of Lemma 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[n + nk - k - \beta] \frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda + \mu)}\ \frac{1}{(n-1)!} |a_n| \le \beta - 1.$$

Now, in view of Lemma 2.6, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n[n + nk - k - \beta] \frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda + \mu)}\ \frac{1}{(n-1)!} |a_n| \\
&\le (\beta - 1) \sum_{n=2}^{\infty} \left[n \frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda + \mu)}\ \frac{1}{(n-1)!} \right] \\
&= (\beta - 1) \left\{ \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda + \mu)}\ \frac{1}{(n-2)!} + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda + \mu)}\ \frac{1}{(n-1)!} \right\} \\
&= (\beta - 1) \left\{ \sum_{n=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda + \mu)}\ \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda + \mu)}\ \frac{1}{(n+1)!} \right\} \\
&= (\beta - 1) \left\{ \left[\mathbb{W}'_{\lambda,\,\mu}(1) - \mathbb{W}_{\lambda,\,\mu}(1) \right] + \left[\mathbb{W}_{\lambda,\,\mu}(1) - 1 \right] \right\} \\
&= (\beta - 1) \left\{ \mathbb{W}'_{\lambda,\,\mu}(1) - 1 \right\} \\
&\le \beta - 1,
\end{aligned}$$

by the given hypothesis. This completes the proof of Theorem 3.4.

Theorem 3.5. *If $\lambda,\ \mu > 0$, for some k $(0 \le k < \infty)$, β $(1 \le \beta < \frac{4+k}{3})$, the* *function $f \in \mathscr{R}^{\tau}(A,\ B)$ and the inequality*

$$(A-B)|\tau|\left\{(1+k)\mathbb{W}_{\lambda,\,\mu}(1)-(k+\beta)\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\left[\mathbb{W}_{\lambda,\,\mu-\lambda}(1)-1\right]+\beta-1\right\}\leq\beta-1 \tag{3.4}$$

is satisfied, then $\Omega(\lambda,\ \mu)f \in k-\mathscr{S}_p^*(\beta)$, where $k-\mathscr{S}_p^*(\beta)$ and $\mathscr{R}^{\tau}(A,\ B)$ are respectively, defined by (1.3) and (1.7), and $\Omega(\lambda,\ \mu)$ is the operator defined by (1.11).

Proof. Let $f \in k-\mathscr{S}^*(\beta)$, where f is of the form (1.4). Since

$$\Omega(\lambda,\ \mu)f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)}\frac{a_n z^n}{(n-1)!}\ .$$

To prove that $\Omega(\lambda,\ \mu)f \in k-\mathscr{S}_p^*(\beta)$, in view of Lemma 2.2, it is sufficient to prove that

$$\sum_{n=2}^{\infty}[n+nk-k-\beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}|a_n|\leq\beta-1.$$

Now, in view of Lemma 2.5, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n+nk-k-\beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}|a_n|\\
&\leq(A-B)|\tau|\left\{\sum_{n=2}^{\infty}\left[(1+k)-\frac{k+\beta}{n}\right]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}\right\}\quad\left(\because\ |a_n|\leq\frac{(A-B)\,|\tau|}{n}\right)\\
&=(A-B)|\tau|\left\{\sum_{n=2}^{\infty}(1+k)\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}-\sum_{n=2}^{\infty}\frac{k+\beta}{n}\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}\right\}\\
&=(A-B)|\tau|\left\{\sum_{n=2}^{\infty}(1+k)\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}-(k+\beta)\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{n!}\right\}\\
&=(A-B)|\tau|\left\{\sum_{n=0}^{\infty}(1+k)\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\ \frac{1}{(n+1)!}-(k+\beta)\sum_{n=0}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\ \frac{1}{(n+2)!}\right\}\\
&=(A-B)|\tau|\left\{\sum_{n=0}^{\infty}(1+k)\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\ \frac{1}{(n+1)!}\right.\\
&\qquad\qquad\left.-(k+\beta)\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(\mu-\lambda)}{\Gamma((n+1)\,\lambda+\mu-\lambda)}\ \frac{1}{(n+1)!}-\frac{\Gamma(\mu-\lambda)}{\Gamma(\mu)}\right]\right\}\\
&=(A-B)|\tau|\left\{(1+k)\left[\mathbb{W}_{\lambda,\,\mu}(1)-1\right]-(k+\beta)\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\left[\mathbb{W}_{\lambda,\,\mu-\lambda}(1)-1\right]+(k+\beta)\right\}\\
&=(A-B)|\tau|\left\{(1+k)\mathbb{W}_{\lambda,\,\mu}(1)-(k+\beta)\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\left[\mathbb{W}_{\lambda,\,\mu-\lambda}(1)-1\right]+\beta-1\right\}\\
&\leq\beta-1,
\end{aligned}$$

by the given hypothesis. This completes the proof of Theorem 3.5.

Theorem 3.6. *If* $\lambda,\ \mu > 0$, *for some* k $(0 \leq k < \infty)$, β $(1 < \beta \leq 2)$, β $(1 \leq \beta < \frac{4+k}{3})$, *the function* $f \in \mathscr{R}^{\tau}(\beta_1)$ *and the inequality*

$$(\beta_1 - 1)\left\{(1+k)\mathbb{W}_{\lambda,\,\mu}(1) - (k+\beta)\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\left[\mathbb{W}_{\lambda,\,\mu-\lambda}(1) - 1\right] + \beta - 1\right\} \leq \beta - 1 \tag{3.5}$$

is satisfied, then $\Omega(\lambda,\ \mu)f \in k - \mathscr{S}_p^*(\beta)$, *where* $k - \mathscr{S}_p^*(\beta)$ *is defined by (1.3), and* $\Omega(\lambda,\ \mu)$ *is the operator defined by (1.11).*

Proof. Let $f \in k - \mathscr{S}^*(\beta)$, where f is of the form (1.4). Since

$$\Omega(\lambda,\ \mu)f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)} \frac{a_n z^n}{(n-1)!}\ .$$

To prove that $\Omega(\lambda,\ \mu)f \in k - \mathscr{S}_p^*(\beta)$, in view of Lemma 2.2, it is sufficient to prove that

$$\sum_{n=2}^{\infty}[n + nk - k - \beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}|a_n| \leq \beta - 1.$$

Now, in view of Lemma 2.5, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n + nk - k - \beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}|a_n| \\
&\leq (\beta_1 - 1)\left\{\sum_{n=2}^{\infty}\left[(1+k) - \frac{k+\beta}{n}\right]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}\right\} \quad \left(\because\ |a_n| \leq \frac{(A-B)\,|\tau|}{n}\right) \\
&= (\beta_1 - 1)\left\{\sum_{n=2}^{\infty}(1+k)\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!} - \sum_{n=2}^{\infty}\frac{k+\beta}{n}\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}\right\} \\
&= (\beta_1 - 1)\left\{\sum_{n=2}^{\infty}(1+k)\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!} - (k+\beta)\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{n!}\right\} \\
&= (\beta_1 - 1)\left\{\sum_{n=0}^{\infty}(1+k)\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\ \frac{1}{(n+1)!} - (k+\beta)\sum_{n=0}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\ \frac{1}{(n+2)!}\right\} \\
&= (\beta_1 - 1)\left\{\sum_{n=0}^{\infty}(1+k)\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\ \frac{1}{(n+1)!}\right. \\
&\qquad\qquad \left. -(k+\beta)\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(\mu-\lambda)}{\Gamma((n+1)\,\lambda+\mu-\lambda)}\ \frac{1}{(n+1)!} - \frac{\Gamma(\mu-\lambda)}{\Gamma(\mu)}\right]\right\}
\end{aligned}$$

$$= (\beta_1 - 1)\left\{(1+k)\left[\mathbb{W}_{\lambda,\,\mu}(1) - 1\right] - (k+\beta)\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\left[\left[\mathbb{W}_{\lambda,\,\mu-\lambda}(1) - 1\right] - \frac{\Gamma(\mu-\lambda)}{\Gamma(\mu)}\right]\right\}$$
$$= (\beta_1 - 1)\left\{(1+k)\mathbb{W}_{\lambda,\,\mu}(1) - (k+\beta)\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\left[\mathbb{W}_{\lambda,\,\mu-\lambda}(1) - 1\right] + \beta - 1\right\}$$
$$\leq \beta - 1,$$

by the given hypothesis. This completes the proof of Theorem 3.6.

Theorem 3.7. *If $\lambda,\ \mu > 0$ for some k $(0 \leq k < \infty)$, β $(1 \leq \beta < \frac{k+4}{3})$, the function $f \in k - \mathscr{PS}_p^*(\beta)$ and the inequality*

$$\mathbb{W}_{\lambda,\,\mu}(1) \leq 2$$

is satisfied, then $\Omega(\lambda,\ \mu)f \in k - \mathscr{S}_p^*(\beta)$, where $k - \mathscr{S}_p^*(\beta)$ and $k - \mathscr{PS}_p^*(\beta)$ are respectively, defined by (1.3) and (1.6), and $\Omega(\lambda,\ \mu)$ is the operator defined by (1.11).

Proof. Let $f \in k - \mathscr{PS}_p^*(\beta)$, where f is of the form (1.4). Since

$$\Omega(\lambda,\ \mu)f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma((n-1)\lambda + \mu)} \frac{a_n z^n}{(n-1)!}\ .$$

To prove that $\Omega(\lambda,\ \mu)f \in k - \mathscr{PS}_p^*(\beta)$, in view of Lemma 2.2, it is sufficient to prove that

$$\sum_{n=2}^{\infty}[n + nk - k - \beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda + \mu)}\ \frac{1}{(n-1)!}|a_n| \leq \beta - 1.$$

Now, in view of Lemma 2.6, we have

$$\sum_{n=2}^{\infty}[n + nk - k - \beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda + \mu)}\ \frac{1}{(n-1)!}|a_n| \quad \left(\because\ |a_n| \leq \frac{\beta - 1}{n + nk - k - \beta}\right)$$
$$\leq (\beta - 1)\sum_{n=2}^{\infty}\left[\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda + \mu)}\ \frac{1}{(n-1)!}\right]$$
$$\leq (\beta - 1)\sum_{n=0}^{\infty}\left[\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda + \mu)}\ \frac{1}{(n+1)!}\right]$$
$$= (\beta - 1)\left[\mathbb{W}_{\lambda,\,\mu}(1) - 1\right]$$
$$\leq \beta - 1,$$

which is true for all $\lambda,\ \mu > 0$. This concludes the proof of Theorem 3.7.

Theorem 3.8. *If $\lambda,\ \mu > 0$, for some k $(0 \leq k < \infty)$, β $(1 \leq \beta < \frac{k+4}{3})$, the function $f \in k-\mathscr{PUCV}^*(\beta)$ and the inequality*

$$\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\left[\mathbb{W}_{\lambda,\,\mu-\lambda}(1)-1\right] \leq 2$$

satisfied, then $\Omega(\lambda,\ \mu)f \in k-\mathscr{S}_p^*(\beta)$, where $k-\mathscr{S}_p^*(\beta)$ and $k-\mathscr{PUCV}^*(\beta)$ are respectively, defined by (1.3) and (1.5), and $\Omega(\lambda,\ \mu)$ is the operator defined by (1.11).

Proof. Let $f \in k-\mathscr{PUCV}^*(\beta)$, where f is of the form (1.4). Since

$$\Omega(\lambda,\ \mu)f(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)}\frac{a_n z^n}{(n-1)!}\ .$$

To prove that $\Omega(\lambda,\ \mu)f \in k-\mathscr{PS}_p^*(\beta)$, in view of Lemma 2.2, it is sufficient to prove that

$$\sum_{n=2}^{\infty}[n+nk-k-\beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}|a_n| \leq \beta-1.$$

Now, in view of Lemma 2.6, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n+nk-k-\beta]\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{(n-1)!}|a_n| \quad \left(\because\ |a_n| \leq \frac{\beta-1}{n[n+nk-k-\beta]}\right)\\
&\leq (\beta-1)\sum_{n=2}^{\infty}\left[\frac{\Gamma(\mu)}{\Gamma((n-1)\,\lambda+\mu)}\ \frac{1}{n!}\right]\\
&\leq (\beta-1)\sum_{n=0}^{\infty}\left[\frac{\Gamma(\mu)}{\Gamma((n+1)\,\lambda+\mu)}\ \frac{1}{(n+2)!}\right]\\
&= (\beta-1)\left\{\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(\mu-\lambda)}{\Gamma((n+1)\,\lambda+\mu-\lambda)}\ \frac{1}{(n+1)!}-\frac{\Gamma(\mu-\lambda)}{\Gamma(\mu)}\right]\right\}\\
&= (\beta-1)\left[\frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\left[\mathbb{W}_{\lambda,\,\mu-\lambda}(1)-1\right]-1\right]\\
&\leq \beta-1,
\end{aligned}$$

which is true for all $\lambda,\ \mu > 0$. This concludes the proof of Theorem 3.8.

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References

[1] Dixit K. K. and Pal S. K., On a class of univalent functions related to complex order, Indian J. Pure Appl. Math., 26, No. 9 (1995), 889-896.

[2] Dixit K. K. and Porwal S., On a certain classes of $k-$uniformly convex functions with negative coefficients, Bull. Cal. Math. Soc., 100 (6) (2008), 639-652.

[3] Gatti N. B., Porwal S. and Magesh N., On certain subclasses of starlike and convex functions associated with Wright function, Advance Studies Contemp. Math., 32(3) (2022), 375-385.

[4] Giri M. K. and Raghavendar K., Inclusion results on hypergeometric functions in a class of analytic functions associated with linear operators, Contemp. Math., 5(2) (2024), 2315-2334.

[5] Gorenflo R., Luchko Y., Mainardi F., Analytic properties and applications of Wright functions, Fract Calc Appl Anal., 2(4), (1999), 383-414.

[6] Kavitha S. and Rajeswari M. R., Inclusion properties involving Gaussian hypergeometric functions for univalent functions with bounded boundary rotation, South East Asian J. Math. Math. Sci., 20(3) (2024), 177-190.

[7] Maharana S., Prajapat J. K. and Bansal D., Geometric properties of Wright function, Math. Bohem., 143(1), (2018), 99-111.

[8] Maharana S. and Sahoo S. K., Inclusion properties of planar harmonic mappings associated with the Wright function, Complex Var. Elliptic Equ., 66(10) (2021), 1619-1641.

[9] Miller S. S and Mocanu P. T., Differential subordinations, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[10] Mondal S. R., Giri M. K., Kondooru R., Sufficient conditions for linear operators related to confluent hypergeometric function and generalized Bessel function of the first kind to belong to a certain class of analytic functions, Symmetry, 16(2024), 662.

[11] Mustafa N., Geometric properties of normalized Wright functions, Math Comput Appl., 21(14), (2016), 1-10.

[12] Porwal S. and Dixit K. K., An application of certain convolution operator involving hypergeometric functions, J. Rajasthan Acad. Phys. Sci., 9(2) (2010), 173–186.

[13] Prajapat J. K., Certain geometric properties of the Wright functions, Integral Transforms Spec Funct., 26(3) (2015), 203-212.

[14] Prakash V., Breaz D., Sivasubramanian S., and El-Deeb S. M., Results of certain subclasses of univalent function related to Bessel functions, Mathematics, 13(2025), 569.

[15] Raina R. K., On univalent and starlike Wright's hypergeometric functions, Rend. Sem. Mat. Univ. Padova, 95 (1996), 11-22.

[16] Sharma P. and Raina R. K., On a generalized hypergeometric operator and its mapping structures, Proc. Jangjeon Math. Soc., 19(3), (2016), 545-562.

[17] Srivastava H. M., Murugusundaramoorthy G. and Sivasubramanian S., Hypergeometric functions in the parabolic starlike and uniformly convex domains, Integral Transforms Spec. Funct., 18, No. 7-8 (2007), 511-520.

[18] Swaminathan A., Certain sufficiency conditions on Gaussian hypergeometric functions, J. Inequal. Pure Appl. Math., 5(4), (2004), 1-10.

[19] Thulasiram T., Suchithra K., Sudharsan T. V. and Murugusundaramoorthy G., Some inclusion results associated with certain subclass of analytic functions inolving Hohlov operator, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 108 No. 2, (2014), 711-720.

[20] Uralegaddi B. A., Ganigi M. D. and Sarangi S. M., Univalent functions with positive coefficients, Tamkang J. Math., 25(3), (1994) 225-230.

[21] Uralegaddi B. A., Ganigi M. D. and Sarangi S. M., Close-to-Convex functions with positive coefficients, Studia Univ. Babes-Balyai, Math., XL(4) (1995), 25–31.

[22] Wright E. M., On the coefficients of power series having exponential singularities, J. Lond. Math. Soc., 8 (1933), 71-79.